

Super-Poincaré Invariant Superstring Field Theory

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Using the topological techniques developed in an earlier paper with Vafa, a field theory action is constructed for any open string with critical $N=2$ worldsheet superconformal invariance. Instead of the Chern-Simons-like action found by Witten, this action resembles that of a Wess-Zumino-Witten model. For the $N=2$ string which describes (2,2) self-dual Yang-Mills, the string field generalizes the scalar field of Yang.

As was shown in recent papers, an $N=2$ string can also be used to describe the Green-Schwarz superstring in a Calabi-Yau background. In this case, one needs three types of string fields which generalize the real superfield of the super-Yang-Mills prepotential, and the chiral and anti-chiral superfields of the Calabi-Yau scalar multiplet. The resulting field theory action for the open superstring in a Calabi-Yau background has the advantages over the standard RNS action that it is manifestly $SO(3,1)$ super-Poincaré invariant and does not require contact terms to remove tree-level divergences.

1. Introduction

The construction of a field theory action for the superstring is an important problem since it may lead to clues about non-perturbative superstring theory which are unobtainable from the on-shell perturbative S-matrix. The standard approach to constructing such an action makes use of the N=1 BRST operator Q , together with the picture-changing operator Z and the inverse-picture-changing operator Y , of the RNS formalism.[1]

The naive equation of motion and gauge-invariance for the Neveu-Schwarz string field A is $QA + ZA^2 = 0$ and $\delta A = Q\Lambda + Z[A, \Lambda]$, where string fields are multiplied using Witten's half-string product and Z is inserted at the vertex midpoint. This naive equation of motion can be obtained from the Cherns-Simons-like action $\int(AQA + \frac{2}{3}ZA^3)$, however gauge invariance requires the addition of quartic and higher-order contact terms[2] to remove the divergence when two Z 's collide.¹

Another disadvantage of the RNS superstring field theory is the lack of manifest spacetime supersymmetry. Besides requiring different string fields for the bosonic and fermionic sectors, the action explicitly involves the picture-changing operators, Z and Y , which do not commute with the spacetime-supersymmetry generators of picture $\pm\frac{1}{2}$.

A different approach to constructing a field theory action for the superstring uses the light-cone Green-Schwarz formalism.[5] This action is manifestly invariant under an $SU(4) \times U(1)$ subgroup of the super-Poincaré group. However because it is completely gauge-fixed, it is difficult to find a geometrical structure. Furthermore, the light-cone Green-Schwarz field theory action suffers from the same problem as the RNS action that contact terms need to be introduced to remove tree-level divergences.[6]

Over the last few years, a new formalism has been developed for the superstring which has critical N=2 worldsheet superconformal invariance.[7] This formalism is manifestly $SO(3,1)$ super-Poincaré invariant and can be used to describe any compactified version of the superstring which contains four-dimensional spacetime-supersymmetry.

Because of the problems with other formalisms, it is natural to try to construct a superstring field theory action using this new formalism. The most obvious approach would be to use the N=2 worldsheet superconformal invariance to construct an N=2 BRST operator and N=2 picture-changing operators, and to look for an N=2 generalization of

¹ Some authors[3] have tried to avoid these divergences by modifying the picture of A so that the action takes the form $\int(AQY^2A + \frac{2}{3}Y^2A^3)$. However this action suffers from the problem that the linearized equation of motion, $QY^2A = 0$, has unphysical solutions when $Y^2A = 0$. [4]

the Chern-Simons-like action. However this approach would probably suffer from the same contact term problems as the other formalisms.

A second approach is to twist the N=2 superconformal generators using the topological techniques developed with Vafa in reference [8], and to construct a new type of string field theory action. As will be shown in this paper, this new type of action resembles a Wess-Zumino-Witten action [9], rather than a Chern-Simons action. The two major advantages of this action are that it is manifestly SO(3,1) super-Poincaré invariant and that it does not require contact terms to remove tree-level divergences.

In section 2 of this paper, the topological description [8] of critical N=2 strings will be reviewed. In this topological description, the BRST operator Q is replaced by two fermionic spin-one generators, G^+ and \tilde{G}^+ , which are constructed entirely out of N=2 matter fields. The linearized equation of motion $QA = 0$ is replaced by $\tilde{G}^+G^+\Phi = 0$ and the linearized gauge invariance $\delta A = Q\Lambda$ is replaced by $\delta\Phi = G^+\Lambda + \tilde{G}^+\bar{\Lambda}$.

In section 3, it will be shown how to construct a string field theory action from this topological description of critical N=2 strings. Whereas Q is covariantized to $Q + A$ in the Chern-Simons-like action, G^+ and \tilde{G}^+ will be covariantized to $e^{-\Phi}G^+e^\Phi$ and \tilde{G}^+ . So instead of the non-linear equation of motion $QA + A^2 = 0$, one finds the equation of motion $\tilde{G}^+(e^{-\Phi}G^+e^\Phi) = 0$, which comes from a Wess-Zumino-Witten type of action. For the critical N=2 string that describes (2,2) self-dual Yang-Mills, it is easy to show that the resulting string field theory action reproduces the action for self-dual Yang-Mills[10] where the string field generalizes the scalar field of Yang[11].

In section 4, the N=2 description of the superstring in a Calabi-Yau background will be reviewed. This N=2 description is obtained by embedding the critical N=1 superstring into a critical N=2 string such that G^+ is the integrand of the N=1 BRST operator and \tilde{G}^+ is the η fermion which comes from fermionizing the N=1 ghosts.[12] N=2 vertex operators are related to N=1 vertex operators by $\Phi = \xi A$, so $\tilde{G}^+G^+\Phi = 0$ implies $QA = 0$. The main advantage of this N=2 description of the superstring is it can be made manifestly super-Poincaré invariant by expressing the RNS variables in terms of GS-like variables. These GS-like variables include the four-dimensional superspace variables x^m , θ^α and $\bar{\theta}^{\dot{\alpha}}$, the conjugate variables for θ^α and $\bar{\theta}^{\dot{\alpha}}$, the chiral boson ρ (similar to the chiral boson ϕ of the N=1 ghost sector), and any $c = 9$ N=2 superconformal field theory.[7]

In section 5, a manifestly super-Poincaré invariant field theory action will be constructed for the superstring. Although this might seem straightforward using results from

the previous sections, there is one difficulty which needs to be overcome. Since the linearized equation of motion $\tilde{G}^+ G^+ \Phi$ has solutions at every $N=1$ picture, the same physical state can be represented by different string fields. However it will be proven that if the string field is restricted to have ρ -charge +1, 0, or -1, each physical state is uniquely represented. The string field with zero ρ -charge will generalize the real superfield that describes the super-Yang-Mills prepotential, while the string fields with ± 1 ρ -charge will generalize the chiral and anti-chiral superfields that describe the Calabi-Yau scalar multiplet. In terms of these three string fields, equations of motion and gauge invariances will be expressed in a manifestly super-Poincaré invariant manner.

Miraculously, an action which yields these equations of motion can be found by comparing with the action of Marcus, Sagnotti, and Siegel for ten-dimensional super-Yang-Mills written in terms of four-dimensional superfields.[13] By promoting their point-particle superfields to string fields, a manifestly $SO(3,1)$ super-Poincaré invariant action will be constructed for any compactification of the open superstring which preserves four-dimensional spacetime-supersymmetry. This superstring field theory action resembles a WZW action and does not require contact terms to remove tree-level divergences.

In section 6, the conclusions of this paper will be summarized and some possible applications will be proposed.

2. Review of the Topological Description of Critical $N=2$ Strings

A critical $N=2$ string contains generators L , G^+ , G^- , and J which satisfy the OPE's of an $N=2$ superconformal algebra with $c = 6$. Since the OPE of $J(y)$ with $J(z)$ goes like $2(y-z)^{-2}$, this $N=2$ algebra can be extended to a small $N=4$ superconformal algebra by defining two new spin-one generators, $J^{++} = e^{\int^z J}$ and $J^{--} = e^{-\int^z J}$, and two new spin- $3/2$ generators, \tilde{G}^+ and \tilde{G}^- , which are defined by taking the pole term in the OPE of J^{++} with G^- and J^{--} with G^+ . Note that G^+ has no singularities with \tilde{G}^- , and $\int G^+$ anticommutes with $\int \tilde{G}^+$ since the OPE of $G^+(y)$ with $\tilde{G}^+(z)$ goes like $\frac{1}{2}(\partial_z - \partial_y)((y-z)^{-1}J^{++}(z))$.

One method for calculating scattering amplitudes for the critical $N=2$ string is to introduce $N=2$ ghosts, construct an $N=2$ BRST operator, and integrate correlation functions of BRST-invariant vertex operators over the moduli of $N=2$ super-Riemann surfaces. However as was shown in a recent paper with Vafa [8], there is a simpler method for calculating scattering amplitudes. After twisting the algebra by the $U(1)$ current J such that

G^+ and \tilde{G}^+ have spin-one while G^- and \tilde{G}^- have spin-two, one can find a convenient choice of N=2 moduli such that the N=2 ghost correlation functions cancel each other out.

After integrating out the N=2 ghost fields, the scattering amplitude can be expressed as an integral over ordinary N=0 moduli of correlation functions of ghost-independent vertex operators. Instead of being contracted with b ghosts, the $3g - 3 + N$ beltrami differentials for the moduli are contracted with the spin-two G^- 's and \tilde{G}^- 's (the relative number of G^- 's and \tilde{G}^- 's depends on the U(1) instanton number of the original N=2 super-Riemann surface).

Although amplitudes for arbitrary genus and U(1) instanton number can be found in reference [8], the most relevant amplitude for open string field theory is the three-point tree amplitude at zero instanton number which is given by:

$$\langle \Phi(z_1)(G^+\Phi(z_2))(\tilde{G}^+\Phi(z_3)) \rangle \quad (2.1)$$

where Φ is a U(1)-neutral vertex operator satisfying $\tilde{G}^+G^+\Phi = 0$ (for unitary strings, this implies that Φ is a weight-zero N=2 primary field), and $G^+\Phi$ signifies the contour integral of spin-one G^+ around Φ . As in all open string theories, the Φ 's carry Chan-Paton factors which will be suppressed throughout this paper.

By deforming the contours of G^+ and \tilde{G}^+ , it is easy to check that this amplitude is symmetric in the three vertices and is invariant under the gauge transformations $\delta\Phi = G^+\Lambda + \tilde{G}^+\bar{\Lambda}$. Note that the contour integral of G^+ anticommutes with the contour integral of \tilde{G}^+ .

The simplest example of a critical N=2 string is the open string which describes (2,2) self-dual Yang-Mills.[10] Its action is

$$\int dz d\bar{z} (\partial_z x_j \bar{\partial}_{\bar{z}} \bar{x}_{\bar{j}} + \psi_{\bar{j}}^- \bar{\partial}_{\bar{z}} \psi_j^+ + \bar{\psi}_{\bar{j}}^- \partial_z \bar{\psi}_j^+), \quad (2.2)$$

and its N=2 superconformal generators are

$$L = \partial_z x_j \partial_z \bar{x}_{\bar{j}} + \psi_{\bar{j}}^- \partial_z \psi_j^+, \quad G^+ = \psi_j^+ \partial_z \bar{x}_{\bar{j}}, \quad G^- = \psi_{\bar{j}}^- \partial_z x_j, \quad J = \psi_j^+ \psi_{\bar{j}}^-$$

where j and \bar{j} take the values 1 or 2. Note that after twisting, ψ_j^+ is spin-zero while $\psi_{\bar{j}}^-$ is spin-one. The additional N=4 superconformal generators are easily found to be

$$\tilde{G}^+ = \epsilon^{jk} \psi_j^+ \partial_z x_k, \quad \tilde{G}^- = \epsilon^{\bar{j}\bar{k}} \psi_{\bar{j}}^- \partial_z \bar{x}_{\bar{k}}, \quad J^{++} = \epsilon^{jk} \psi_j^+ \psi_k^+, \quad J^{--} = \epsilon^{\bar{j}\bar{k}} \psi_{\bar{j}}^- \psi_{\bar{k}}^-.$$

Up to gauge transformations, the only momentum-dependent U(1)-neutral vertex operator satisfying $\tilde{G}^+G^+\Phi = 0$ is $\Phi = \exp(ik_j \bar{x}_{\bar{j}} + i\bar{k}_{\bar{j}} x_j)$ where $k_j \bar{k}_{\bar{j}} = 0$. After performing the correlation functions over the N=2 matter fields (note that ψ_j^+ has a zero mode), one finds that (2.1) produces the usual three-point tree amplitude $\bar{k}_{\bar{j}}^2 k_j^3 f^{I_1 I_2 I_3}$ where $f^{I_1 I_2 I_3}$ is the structure constant for the Chan-Paton factors.

3. String Field Theory for Open N=2 Strings

In order to construct a string field theory action, it is natural to generalize the on-shell vertex operator to an off-shell string field Φ which is a U(1)-neutral function of $x(\sigma)$ and $\psi(\sigma)$ (note that Φ is bosonic, in contrast with the Neveu-Schwarz string field, A , which is fermionic). Since complex conjugation flips $J \rightarrow -J$ and therefore changes the sign of the twist, the reality condition on Φ will be

$$(\overline{\Phi(\sigma)})^R = \Phi(\pi - \sigma) \quad (3.1)$$

where the bar signifies hermitian conjugation, and the R signifies an SU(2) rotation which returns the original twist by transforming $J \rightarrow -J$, $J^{++} \rightarrow J^{--}$, and $J^{--} \rightarrow J^{++}$. For an N=2 primary field Ψ of U(1)-charge m , $(\Psi)^R$ is the pole of order m^2 in the OPE of Ψ with $e^{-m \int^z J}$ (e.g., for the self-dual N=2 string, $(\psi_j^+)^R = \epsilon_{jk} \psi_k^-$).

For the string field theory action to be correct, the quadratic term in the action should enforce the linearized equation of motion $\tilde{G}^+ G^+ \Phi = 0$, while the cubic term should produce the correct on-shell three-point amplitude. Finally, the action should contain a gauge invariance whose linearized form is $\delta\Phi = G^+ \Lambda + \tilde{G}^+ \bar{\Lambda}$.

The quadratic and cubic terms in the action are easily found to be of the form

$$\int \left((G^+ \Phi)(\tilde{G}^+ \Phi) + \Phi \{ G^+ \Phi, \tilde{G}^+ \Phi \} \right),$$

however there is no non-linear version of the gauge transformation $\delta\Phi = G^+ \Lambda + \tilde{G}^+ \bar{\Lambda}$ which leaves this action invariant. This means that quartic and higher-order terms need to be added, which should not be surprising since the non-linear equation of motion for (2,2) self-dual Yang-Mills is $\partial_j (e^{-\phi} \bar{\partial}_{\bar{j}} e^\phi) = 0$, where $A_j = e^{\frac{1}{2}\phi} \partial_j e^{-\frac{1}{2}\phi}$ and $A_{\bar{j}} = e^{-\frac{1}{2}\phi} \bar{\partial}_{\bar{j}} e^{\frac{1}{2}\phi}$ are the self-dual Yang-Mills gauge fields[11]. Note however that the quartic and higher-order terms will not have infinite coefficients like those of the RNS field theory action, and will be completely explicit functions of the string fields.

The obvious guess for the non-linear generalization of $\tilde{G}^+ G^+ \Phi = 0$ is therefore $\tilde{G}^+ (e^{-\Phi} G^+ e^\Phi) = 0$ where multiplication of string fields is always performed using Witten's half-string overlap. If ϕ is the component of Φ which depends only on the zero mode of x , then $\tilde{G}^+ (e^{-\Phi} G^+ e^\Phi)$ contains the term $\epsilon^{kl} \psi_k^+ \partial_l (e^{-\phi} \psi_j^+ \bar{\partial}_{\bar{j}} e^\phi) = (\frac{1}{2} \epsilon^{kl} \psi_k^+ \psi_l^+) \partial_j (e^{-\phi} \bar{\partial}_{\bar{j}} e^\phi)$.

The action which produces this equation of motion is a straightforward generalization of the WZW model where the two-dimensional derivatives ∂_z and $\bar{\partial}_{\bar{z}}$ are replaced by G^+ and \tilde{G}^+ . The string field theory action is

$$\frac{1}{2} \int \left((e^{-\Phi} G^+ e^\Phi) (e^{-\Phi} \tilde{G}^+ e^\Phi) - \int_0^1 dt (e^{-t\Phi} \partial_t e^{t\Phi}) \{e^{-t\Phi} G^+ e^{t\Phi}, e^{-t\Phi} \tilde{G}^+ e^{t\Phi}\} \right). \quad (3.2)$$

Note that this action is real since $\overline{(e^{-\Phi} G^+ e^\Phi)}^R = (\tilde{G}^+ e^\Phi) e^{-\Phi}$. In addition to producing the correct linearized equations of motion and three-point tree amplitude, this action contains the non-linear gauge invariance,

$$\delta e^\Phi = (G^+ \Lambda) e^\Phi + e^\Phi (\tilde{G}^+ \bar{\Lambda}), \quad (3.3)$$

which generalizes the linearized gauge invariance $\delta\Phi = G^+ \Lambda + \tilde{G}^+ \bar{\Lambda}$.

4. Review of the N=2 Description of the Superstring

As was described in reference [12], an N=2 description of the superstring can be obtained by “embedding” the critical N=1 string into a critical N=2 string. The resulting N=4 superconformal generators (after twisting) consist of

$$L = L_{RNS}, \quad G^+ = j_{BRST}, \quad \tilde{G}^+ = \eta, \quad G^- = b, \quad \tilde{G}^- = bZ, \\ J^{++} = c\eta, \quad J = cb + \eta\xi, \quad J^{--} = b\xi, \quad (4.1)$$

where L_{RNS} is the RNS stress-energy tensor (including the N=1 ghosts), $Q_{RNS} = \int j_{BRST}$, $Z = \{Q, \xi\}$, and the N=1 ghosts are fermionized as $\gamma = \eta e^\phi$ and $\beta = (\partial_z \xi) e^{-\phi}$. [8] Note that the “large” hilbert space is necessary for the N=2 description since the zero mode of ξ explicitly appears in the superconformal generators.

As in the N=2 string for (2,2) self-dual Yang-Mills, the physical vertex operators for the superstring satisfy $\tilde{G}^+ G^+ \Phi = 0$ and are subject to the gauge invariances $\delta\Phi = G^+ \Lambda + \tilde{G}^+ \bar{\Lambda}$ (note that the N=2 vertex operator Φ is related to the N=1 vertex operator A by $\Phi = \xi A$, so $QA = 0$ implies that $\tilde{G}^+ G^+ \Phi = 0$). However a crucial new feature for the superstring is that G^+ and \tilde{G}^+ have trivial cohomology (i.e., $G^+ \Phi = 0$ implies that $\Phi = G^+ (\xi Y \Phi)$ and $\tilde{G}^+ \Phi = 0$ implies that $\Phi = \tilde{G}^+ (\xi \Phi)$). As was shown in reference [8], this means that each physical state of the superstring is represented by an infinite ladder

of vertex operators, V_n , where n labels the N=1 picture. Adjacent steps on the ladder are related by $G^+ V_n = c_n \tilde{G}^+ V_{n+1}$ for some constant c_n .

The main advantage of the N=2 description of the superstring is that it can be made manifestly SO(3,1) super-Poincaré invariant for any compactification which preserves four-dimensional spacetime-supersymmetry (this is not possible in the N=1 description since the N=1 fermionic generator is not GSO projected). By finding a field redefinition from RNS variables to GS variables, all superconformal generators and vertex operators can be expressed in manifestly super-Poincaré invariant notation. This field redefinition takes the four-dimensional RNS variables x^m , ψ^m , c , b , ξ , η , and ϕ into the four-dimensional GS variables x^m , θ^α , $\bar{\theta}^{\dot{\alpha}}$, p_α , $\bar{p}_{\dot{\alpha}}$, and ρ (p_α and $\bar{p}_{\dot{\alpha}}$ are conjugates to θ^α and $\bar{\theta}^{\dot{\alpha}}$, and ρ is a chiral boson similar to ϕ of the N=1 ghost sector). As in the RNS formalism, the internal six-dimensional variables of the GS superstring will be described by a $c = 9$ N=2 superconformal field theory.

Under this field redefinition, the twisted N=2 superconformal generators of equation (4.1) get mapped into the following generators:

$$L = L_4 + L_6, \quad G^+ = G_4^+ + G_6^+, \quad G^- = G_4^- + G_6^-, \quad J = J_4 + J_6$$

where

$$\begin{aligned} L_4 &= \frac{1}{2} \partial_z x^m \partial_z x_m + p_\alpha \partial_z \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_z \bar{\theta}^{\dot{\alpha}} + \frac{1}{2} \partial_z \rho \partial_z \rho + \frac{1}{2} \partial^2 \rho \\ G_4^+ &= e^\rho (d)^2, \quad G_4^- = e^{-\rho} (\bar{d})^2, \quad J_4 = -\partial_z \rho, \end{aligned} \quad (4.2)$$

$d_\alpha = p_\alpha + i\bar{\theta}^{\dot{\alpha}} \partial_z x_{\alpha\dot{\alpha}} - \frac{1}{2}(\bar{\theta})^2 \partial_z \theta_\alpha + \frac{1}{4}\theta_\alpha \partial_z (\bar{\theta})^2$, $\bar{d}_{\dot{\alpha}} = \bar{p}_{\dot{\alpha}} + i\theta^\alpha \partial_z x_{\alpha\dot{\alpha}} - \frac{1}{2}(\theta)^2 \partial_z \bar{\theta}_{\dot{\alpha}} + \frac{1}{4}\bar{\theta}_{\dot{\alpha}} \partial_z (\theta)^2$, $(d)^2$ means $\epsilon^{\alpha\beta} d_\alpha d_\beta$, and $[L_6, G_6^+, G_6^-, J_6]$ form a $c = 9$ N=2 superconformal field theory.

The remaining N=4 superconformal generators can be constructed from these N=2 generators in the usual way. The only such generator which will be needed for the string field theory is $\tilde{G}^+ = \tilde{G}_4^+ + \tilde{G}_6^+$, where

$$\tilde{G}_4^+ = e^{\int^z J_6} e^{-2\rho} (\bar{d})^2, \quad \tilde{G}_6^+ = e^{-\rho} \tilde{G}_6^{++},$$

and G_6^{++} is the pole term in the OPE of G_6^- and $e^{\int^z J_6}$.

These generators are manifestly invariant under the four-dimensional spacetime-supersymmetry generated by $q_\alpha = \int dz (p_\alpha - i\bar{\theta}^{\dot{\alpha}} \partial_z x_{\alpha\dot{\alpha}} - \frac{1}{4}(\bar{\theta})^2 \partial_z \theta_\alpha)$ and $\bar{q}_{\dot{\alpha}} = \int dz (\bar{p}_{\dot{\alpha}} - i\theta^\alpha \partial_z x_{\alpha\dot{\alpha}} - \frac{1}{4}(\theta)^2 \partial_z \bar{\theta}_{\dot{\alpha}})$, which satisfy the relation $\{q_\alpha, \bar{q}_{\dot{\alpha}}\} = -2i \int dz \partial_z x_{\alpha\dot{\alpha}}$. Note that the RNS picture operator, $\int dz (\partial_z \phi - \xi \eta)$, is mapped into $\int dz (\partial_z \rho + \frac{1}{2}(p_\alpha \theta^\alpha - \bar{p}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}))$, so q_α is in the $+\frac{1}{2}$ picture while $\bar{q}_{\dot{\alpha}}$ is in the $-\frac{1}{2}$ picture.

In certain pictures, vertex operators for the massless fields take a particularly simple form when written in terms of the GS variables. The vertex operator for the four-dimensional super-Yang-Mills multiplet is $\Phi = v(x, \theta, \bar{\theta})$ where v is the real superfield for the prepotential, while the vertex operators for the Calabi-Yau scalar multiplet are $\Phi = (\bar{\theta})^2 e^\rho \bar{\Psi}^j \omega_j(x, \theta)$ and $\Phi = (\theta)^2 e^{-\rho} \Psi_j \bar{\omega}^j(x, \bar{\theta})$ where Ψ_j and $\bar{\Psi}^j$ are the chiral and anti-chiral primary fields for the internal $c = 9$ N=2 superconformal field theory, and ω_j and $\bar{\omega}^j$ are chiral and anti-chiral superfields for the scalar multiplet.

5. Open Superstring Field Theory

5.1. Construction of the Three String Fields

Given the results from the previous sections, an obvious guess for constructing a superstring field theory action is to generalize the on-shell vertex operator Φ to an off-shell string field. However this would be incorrect since, as was shown in the previous section, each physical state is represented by an infinite number of vertex operators satisfying $\tilde{G}^+ G^+ \Phi = 0$. Although one could restrict all bosonic vertex operators to be in the zero picture and all fermionic vertex operators to be in the $-\frac{1}{2}$ picture, this would break the manifest spacetime supersymmetry.

The way to avoid this problem is to realize that although $\tilde{G}^+ G^+ \Phi = 0$ has infinitely many solutions, only one such solution satisfies the more restrictive equation

$$(G^+ + \tilde{G}^+) \hat{\Phi} = 0. \quad (5.1)$$

It is easily shown that up to an overall constant, this solution is $\hat{\Phi} = \sum_{n=-\infty}^{\infty} V_n$ where V_n are eigenvectors of the picture operator which satisfy $G^+ V_n = -\tilde{G}^+ V_{n+1}$. Note that $(G^+ + \tilde{G}^+)^2 = 0$, so the solution contains the gauge transformations

$$\delta \hat{\Phi} = (G^+ + \tilde{G}^+) \hat{\Lambda}. \quad (5.2)$$

Since the picture operator does not commute with spacetime supersymmetry, it will be more convenient to expand $\hat{\Phi}$ in eigenvectors of the $\int \partial \rho$ operator as

$$\hat{\Phi} = \sum_{n=-\infty}^{\infty} \Phi_n \quad (5.3)$$

where $\Phi_n = e^{n\rho} F_n$ and F_n carries Calabi-Yau charge $-n$. Note that $\hat{\Phi}$ satisfies the reality condition of equation (3.1), so $(\overline{\Phi_n(\sigma)})^R = \Phi_{-n}(\pi - \sigma)$.

From equation (5.1), the Φ_n 's must satisfy

$$G_4^+ \Phi_n + G_6^+ \Phi_{n+1} + \tilde{G}_6^+ \Phi_{n+2} + \tilde{G}_4^+ \Phi_{n+3} = 0 \quad (5.4)$$

since $(G_4^+, G_6^+, \tilde{G}_6^+, \tilde{G}_4^+)$ carry ρ -charge $(1, 0, -1, -2)$. Under the gauge transformations parameterized by $\hat{\Lambda} = \sum_{n=-\infty}^{\infty} \Lambda_n$ where Λ_n carries ρ -charge n ,

$$\delta\Phi_n = G_4^+ \Lambda_{n-1} + G_6^+ \Lambda_n + \tilde{G}_6^+ \Lambda_{n+1} + \tilde{G}_4^+ \Lambda_{n+2}. \quad (5.5)$$

It will now be proven that up to gauge transformations, all Φ_n 's can be expressed in terms of Φ_{-1} , Φ_0 , and Φ_1 . The proof is inductive in m . Suppose that Φ_p is known for $|p| \leq m$ where m is positive. Then for equation (5.4) where $n = -m-1$ or $n = m-2$, one can solve for $G_4^+ \Phi_{-m-1}$ and $\tilde{G}_4^+ \Phi_{m+1}$ in terms of the Φ_p 's. But under the gauge transformations parameterized by Λ_{-m-2} and Λ_{m+3} , $\delta\Phi_{-m-1} = G_4^+ \Lambda_{-m-2}$, $\delta\Phi_{m+1} = \tilde{G}_4^+ \Lambda_{m+3}$, and all Φ_p 's for $|p| \leq m$ are left unchanged. Since G_4^+ and \tilde{G}_4^+ have trivial cohomology (note that $G_4^+(e^{-\rho}(\theta)^2) = 1$ and $\tilde{G}_4^+(e^{2\rho} - \int^z J_6(\bar{\theta})^2) = 1$), this means that Φ_{-m-1} and Φ_{m+1} can be expressed in terms of the Φ_p 's up to a gauge transformation.

So any solution to $(G^+ + \tilde{G}^+) \hat{\Phi} = 0$ can be expressed in terms of the three string fields Φ_{-1} , Φ_0 , and Φ_1 . By analyzing equation (5.4) for $n = -1, 0$, and 1 , it is easy to check that the remaining three string fields must satisfy the linearized equations of motion:

$$\tilde{G}_4^+ G_4^+ \Phi_{-1} + \tilde{G}_4^+ G_6^+ \Phi_0 + \tilde{G}_4^+ \tilde{G}_6^+ \Phi_1 = 0, \quad (5.6)$$

$$(\tilde{G}_6^+ G_6^+ + \tilde{G}_4^+ G_4^+) \Phi_0 + \tilde{G}_4^+ G_6^+ \Phi_1 + \tilde{G}_6^+ G_4^+ \Phi_{-1} = 0,$$

$$G_4^+ G_6^+ \Phi_{-1} + G_4^+ \tilde{G}_6^+ \Phi_0 + G_4^+ \tilde{G}_4^+ \Phi_1 = 0.$$

These equations of motion are invariant under the six independent linearized gauge transformations:

$$\delta\Phi_{-1} = G_4^+ \Lambda_{-2} + G_6^+ \Lambda_{-1} + \tilde{G}_6^+ \Lambda_0 + \tilde{G}_4^+ \Lambda_1, \quad (5.7)$$

$$\delta\Phi_0 = G_4^+ \Lambda_{-1} + G_6^+ \Lambda_0 + \tilde{G}_6^+ \Lambda_1 + \tilde{G}_4^+ \Lambda_2,$$

$$\delta\Phi_1 = G_4^+ \Lambda_0 + G_6^+ \Lambda_1 + \tilde{G}_6^+ \Lambda_2 + \tilde{G}_4^+ \Lambda_3.$$

So up to these gauge transformations, there is a one-to-one correspondence between physical states of the superstring and fields Φ_{-1} , Φ_0 , Φ_1 satisfying equation (5.6).

5.2. Construction of the Superstring Field Theory Action

In order to construct a superstring field theory action, one needs to find a non-linear generalization of the equations of motion and gauge invariances from (5.6) and (5.7). Miraculously, one can guess this non-linear generalization by analyzing the paper of Marcus, Sagnotti, and Siegel on ten-dimensional super-Yang-Mills written in terms of four-dimensional superfields.[13]

In this paper, it was found useful to covariantize the four-dimensional superspace derivatives as

$$\nabla_\alpha = e^{-v} D_\alpha e^v, \quad \bar{\nabla}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}, \quad (5.8)$$

and the six-dimensional spacetime derivatives as

$$\bar{\nabla}^j = e^{-v} (\bar{\partial}^j + \bar{\omega}^j) e^v, \quad \nabla_j = \partial_j - \omega_j,$$

where v is the real superfield which describes the four-dimensional part of the gauge field, ω_j and $\bar{\omega}^j$ are the chiral and anti-chiral superfields which describe the six-dimensional part of the gauge field ($j = 1$ to 3), and $\nabla_A = D_A - i\Gamma_A$ where Γ_A is the super-connection (note that $\omega_j = i\Gamma_j$ and $e^{-v}\bar{\omega}^j e^v = -i\Gamma^j$).

These covariant derivatives satisfy the identities

$$F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = F_\alpha^j = F_{\dot{\alpha}j} = 0 \quad (5.9)$$

where $F_{AB} = \{\nabla_A, \nabla_B\}$, and transform as $\delta\nabla_A = [\nabla_A, \sigma]$ under the gauge-transformation

$$\delta e^v = \bar{\sigma} e^v + e^v \sigma, \quad \delta\omega_j = -\partial_j \sigma + [\omega_j, \sigma], \quad \delta\bar{\omega}^j = -\bar{\partial}^j \bar{\sigma} - [\bar{\omega}^j, \bar{\sigma}], \quad (5.10)$$

where $\bar{\sigma} = (D)^2 \lambda$ and $\sigma = (\bar{D})^2 \bar{\lambda}$ for some λ and $\bar{\lambda}$.

In terms of these four-dimensional superfields, the non-linear equations of motion for ten-dimensional super-Yang-Mills are:

$$2\{\nabla^\alpha, W_\alpha\} = F_j^j, \quad 2\{\nabla^\alpha, F_{\alpha j}\} = \epsilon_{jkl} F^{kl}, \quad 2\{\bar{\nabla}^{\dot{\alpha}}, F_{\dot{\alpha}}^j\} = \epsilon^{jkl} F_{kl}, \quad (5.11)$$

where $W_\alpha = [\bar{\nabla}^{\dot{\alpha}}, \{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\}]$ is the four-dimensional chiral field strength.

Recall that the vertex operator for the super-Yang-Mills multiplet is simply $\Phi = v$, while for the Calabi-Yau massless multiplet, it is $\Phi = e^\rho(\bar{\theta})^2 \bar{\Psi}^j \omega_j$ and $\Phi = e^{-\rho}(\theta)^2 \Psi_j \bar{\omega}^j$. The massless ten-dimensional super-Yang-Mills fields are therefore described by $\Phi_{-1} = e^{-\rho}(\theta)^2 \psi_j \bar{\omega}^j$, $\Phi_0 = v$, and $\Phi_1 = e^\rho(\bar{\theta})^2 \bar{\psi}^j \omega_j$.

Since $G_4^+ = e^\rho(d)^2$ and $\tilde{G}_4^+ = \frac{1}{6}e^{-2\rho}\epsilon^{jkl}\psi_j\psi_k\psi_l(\bar{d})^2$ in a flat background, it is natural to covariantize these operators to

$$\mathcal{G}_4^+ = e^{-V}G_4^+e^V, \quad \tilde{\mathcal{G}}_4^+ = \tilde{G}_4^+ \quad (5.12)$$

where $V \equiv \Phi_0$ (so for massless excitations, $d_\alpha \rightarrow e^{-v}d_\alpha e^v$ and $\bar{d}_{\dot{\alpha}} \rightarrow \bar{d}_{\dot{\alpha}}$). Similarly, since $G_6^+ = \psi_j\partial_z\bar{x}^j$ and $\tilde{G}_6^+ = \frac{1}{2}e^{-\rho}\epsilon^{jkl}\psi_j\psi_k\partial_zx_l$ in a flat background, it is natural to covariantize these operators to

$$\mathcal{G}_6^+ = e^{-V}(G_6^+ + \bar{\Omega})e^V \quad \tilde{\mathcal{G}}_6^+ = \tilde{G}_6^+ - \Omega$$

where $\bar{\Omega} \equiv G_4^+\Phi_{-1}$ and $\Omega \equiv \tilde{G}_4^+\Phi_1$ (so for massless excitations, $\partial_z\bar{x}^j \rightarrow \partial_z\bar{x}^j - \bar{\omega}^j$ and $\partial_zx_j \rightarrow \partial_zx_j + \omega_j$).

Like their point-particle counterparts in (5.9), these covariantized operators satisfy the identities:

$$\{\mathcal{G}_4^+, \mathcal{G}_4^+\} = \{\tilde{\mathcal{G}}_4^+, \tilde{\mathcal{G}}_4^+\} = \{\mathcal{G}_4^+, \mathcal{G}_6^+\} = \{\tilde{\mathcal{G}}_4^+, \tilde{\mathcal{G}}_6^+\} = 0 \quad (5.13)$$

and transform as $\delta\mathcal{G}_A = [\mathcal{G}_A, \Sigma]$ under the gauge transformations

$$\delta e^V = \bar{\Sigma}e^V + e^V\Sigma, \quad \delta\Omega = -\tilde{G}_6^+\Sigma + [\Omega, \Sigma], \quad \delta\bar{\Omega} = -G_6^+\bar{\Sigma} - [\bar{\Omega}, \bar{\Sigma}] \quad (5.14)$$

where $\bar{\Sigma} = G_4^+\Lambda_{-1}$ and $\Sigma = \tilde{G}_4^+\Lambda_2$.

A natural string generalization of the point-particle equations of motion in equation (5.11) is

$$\{\mathcal{G}_4^+, \tilde{\mathcal{G}}_6^+\} = -\{\mathcal{G}_6^+, \tilde{\mathcal{G}}_6^+\}, \quad (5.15)$$

$$2\{\mathcal{G}_4^+, \tilde{\mathcal{G}}_6^+\} = -\{\mathcal{G}_6^+, \mathcal{G}_6^+\}, \quad 2\{\tilde{\mathcal{G}}_4^+, \mathcal{G}_6^+\} = -\{\tilde{\mathcal{G}}_6^+, \tilde{\mathcal{G}}_6^+\}.$$

These equations can be combined with the identities of equation (5.13) to imply that

$$(\mathcal{G}_4^+ + \mathcal{G}_6^+ + \tilde{\mathcal{G}}_4^+ + \tilde{\mathcal{G}}_6^+)^2 = 0, \quad (5.16)$$

which is the natural generalization of $(Q + A)^2 = 0$ for the Chern-Simons-like action.

In addition to the gauge invariances of equation (5.14), the equations of motion implied by (5.16) are also invariant under the following gauge transformations:

$$\delta e^V = e^V(\mathcal{G}_6^+\Lambda_0 + \tilde{\mathcal{G}}_6^+\Lambda_1), \quad (5.17)$$

$$\delta\Omega = \tilde{G}_4^+(\mathcal{G}_4^+\Lambda_0 + \mathcal{G}_6^+\Lambda_1), \quad \delta\bar{\Omega} = G_4^+(e^V(\tilde{\mathcal{G}}_6^+\Lambda_0 + \tilde{\mathcal{G}}_4^+\Lambda_1)e^{-V}).$$

Unlike the gauge transformations of equation (5.14), these gauge transformations have no super-Yang-Mills counterpart since there is no massless contribution to Λ_0 or Λ_1 .

Finally, a superstring field theory action which yields these equations of motion can be constructed by comparing with the following point-particle action of reference [13] for ten-dimensional super-Yang-Mills:

$$\begin{aligned} & \frac{1}{2} \int d^{10}x [-2 \int d^2\theta W^\alpha W_\alpha \\ & + \int d^4\theta \left((e^{-v} \bar{\partial}^j e^v)(e^{-v} \partial_j e^v) - \int_0^1 dt (e^{-tv} \partial_t e^{tv}) \{e^{-tv} \bar{\partial}^j e^{tv}, e^{-tv} \partial_j e^{tv}\} \right) \\ & + 2 \int d^4\theta ((\partial_j e^{-v}) \bar{\omega}^j e^v + e^v \omega_j (\bar{\partial}^j e^{-v}) + e^{-v} \bar{\omega}^j e^v \omega_j) \\ & + \int d^2\theta \epsilon^{jkl} (\omega_j \partial_k \omega_l + \frac{2}{3} \omega_j \omega_k \omega_l) + \int d^2\bar{\theta} \epsilon_{jkl} (\bar{\omega}^j \bar{\partial}^k \bar{\omega}^l - \frac{2}{3} \bar{\omega}^j \bar{\omega}^k \bar{\omega}^l)]. \end{aligned} \quad (5.18)$$

The superstring generalization of this action is:

$$\begin{aligned} & \frac{1}{2} \int [(e^{-V} G_4^+ e^V)(e^{-V} \tilde{G}_4^+ e^V) - \int_0^1 dt (e^{-tV} \partial_t e^{tV}) \{e^{-tV} G_4^+ e^{tV}, e^{-tV} \tilde{G}_4^+ e^{tV}\} \\ & + (e^{-V} G_6^+ e^V)(e^{-V} \tilde{G}_6^+ e^V) - \int_0^1 dt (e^{-tV} \partial_t e^{tV}) \{e^{-tV} G_6^+ e^{tV}, e^{-tV} \tilde{G}_6^+ e^{tV}\} \\ & + 2 \left((\tilde{G}_6^+ e^{-V}) \bar{\Omega} e^V + e^V \Omega (G_6^+ e^{-V}) + e^{-V} \bar{\Omega} e^V \Omega \right) \\ & - (\Omega \tilde{G}_6^+ \Phi_1 - \frac{2}{3} \Omega \Omega \Phi_1) + (\bar{\Omega} G_6^+ \Phi_{-1} + \frac{2}{3} \bar{\Omega} \bar{\Omega} \Phi_{-1})]. \end{aligned} \quad (5.19)$$

The only subtle part of this generalization is that unlike the point-particle action, a WZW term is used for both the four-dimensional and six-dimensional parts of the string action for V . Note also that chiral F -terms in the point-particle action are annihilated by \tilde{G}_4^+ in the string action and, because \tilde{G}_4^+ has trivial cohomology, can be turned into D -terms by pulling \tilde{G}_4^+ off one of the Ω 's. Since \tilde{G}_4^+ is the inverse of the ξ zero mode, turning F -terms into D -terms is like going from the small to the large RNS hilbert space.

To show that this superstring field theory action is correct, one should check that its linearized equations of motion and gauge invariances reproduce the on-shell conditions of physical vertex operators, and that the cubic term in the action produces the three-point tree-level scattering amplitude.

It is straightforward to show that the linearized part of (5.15), (5.14), and (5.17), reproduce (5.6)and (5.7)(the Λ_{-2} and Λ_3 gauge transformations act trivially on the string fields), and therefore define the on-shell conditions for the physical vertex operator $\hat{\Phi}$ which is constructed out of Φ_{-1} , Φ_0 , and Φ_1 .

The cubic term in the superstring field theory action of (5.19)is:

$$\begin{aligned} & \frac{1}{2} \int \left[\frac{-1}{3} V(\{G_4^+ V, \tilde{G}_4^+ V\} + \{G_6^+ V, \tilde{G}_6^+ V\}) \right. \\ & \quad \left. - V(\{\tilde{G}_6^+ V, \bar{\Omega}\} + \{G_6^+ V, \bar{\Omega}\} + 2\{\Omega, \bar{\Omega}\}) + \frac{2}{3} (\Omega\bar{\Omega}\Phi_1 - \bar{\Omega}\bar{\Omega}\Phi_{-1}) \right]. \end{aligned} \quad (5.20)$$

Suppose that the three-point amplitude involves vertex operators, $\Phi(z_r)$, which contain no Calabi-Yau charge in the zero picture. Then $V(z_r) = \Phi(z_r)$ and $\Omega(z_r) = \bar{\Omega}(z_r) = 0$. It is easy to check that in this case, (5.20)reproduces the three-point amplitude of (2.1). But since the coefficients in (5.20)are restricted by the requirement of on-shell gauge invariance under the transformations of (5.7), this is enough to prove that (5.20)gives the correct three-point amplitude even when Ω and $\bar{\Omega}$ are non-zero.

6. Conclusion

In this paper, a WZW-like field theory action was constructed for open strings with critical N=2 superconformal invariance. For the N=2 string which describes (2,2) self-dual Yang-Mills, this field theory action generalizes the point-particle action for the scalar field of Yang. For the N=2 string which describes the superstring in a Calabi-Yau background, the action generalizes the point-particle action for ten-dimensional super-Yang-Mills written in terms of four-dimensional superfields.

In both of these string field theory actions, only the U(1)-neutral string fields were considered, which correspond to the matter sector of the field theory. However it should also be possible to consider U(1)-charged string fields which represent the ghost sector of the field theory. Studying the ghost contributions to these actions would be interesting since for (2,2) self-dual Yang-Mills, there is a conjecture that topological supersymmetry relates the matter and ghost sectors.[8]Furthermore, there is a conjecture that adding four bosonic and four fermionic degrees of freedom to the light-cone is useful for constructing covariant actions of supersymmetric field theories.[14] Although the superstring field theory action in this paper involves adding three bosons (x^+ , x^- , and ρ) and six fermions ($\bar{\theta}^{\dot{\alpha}}$, $\bar{p}_{\dot{\alpha}}$,

θ^2 , and p_2) to the light-cone, two of the fermions could be bosonized to produce a fourth boson.

A second important result of this paper was the discovery that all physical states of the superstring are uniquely represented by three string fields, V , Ω , and $\bar{\Omega}$, which generalize the real, chiral, and anti-chiral superfields of four-dimensional superspace. In terms of V , Ω , and $\bar{\Omega}$, it was straightforward to generalize from the ten-dimensional super-Yang-Mills action to an open superstring field theory action. Maybe these three string fields can be used to generalize other properties of four-dimensional supersymmetric field theories (perhaps even non-perturbative properties) from point-particle language to the superstring.

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